Topology and its Applications 157 (2010) 522–529

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Primitive elements in rings of continuous functions

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ARTICLE INFO

Article history:
Received 12 November 2008
Received in revised form 16 October 2009
Accepted 17 October 2009

MSC:
primary 54C40
secondary 13B21, 46E25

Keywords:
Rings of continuous functions
Primitive element
Finite extension
Integral extension
Principal ideal
Monic polynomial
Covering
Trivial covering

ABSTRACT

Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. The map π induces, by composition, an injective morphism $C(Y) \to C(X)$ between the corresponding rings of real-valued continuous functions, and this morphism allows us to consider $C(Y)$ as a subring of $C(X)$. This paper deals with algebraic properties of the ring extension $C(Y) \subseteq C(X)$ in relation to topological properties of the map $\pi : X \to Y$. We prove that if the extension $C(Y) \subseteq C(X)$ has a primitive element, i.e., $C(X) = C(Y)[f]$, then it is a finite extension and, consequently, the map π is locally injective. Moreover, for each primitive element f we consider the ideal $I_f = \{P(t) \in C(Y)[t]\colon\ P(f) = 0\}$ and prove that, for a connected space Y, I_f is a principal ideal if and only if $\pi : X \to Y$ is a trivial covering.

0. Introduction

Every continuous map $X \to Y$ defines, by composition, a morphism $C(Y) \to C(X)$ between the corresponding rings of real-valued continuous functions. This morphism defines in $C(X)$ a natural structure of $C(Y)$-module. When both structures, ring and $C(Y)$-module, are considered together in $C(X)$, it is said that $C(X)$ is a $C(Y)$-algebra. Our general purpose is to determine the algebraic properties of the morphism $C(Y) \to C(X)$ in terms of the topological properties of the map $X \to Y$.

L. Childs has established in [3] the correspondence between finite unbranched coverings of a topological space Y (see Definition 3.1 below) and separable $C(Y)$-algebras that are finitely generated projective $C(Y)$-modules. M.A. Mulero has proved in [9] the going-up and going-down theorems for the morphism $C(Y) \to C(X)$ defined by an open and closed map $X \to Y$, and she has established in [10] the relationship between finite (branched) coverings (open and closed, finite to one, continuous maps) $X \to Y$ and integral and flat morphisms $C(Y) \to C(X)$.

Another particular case consists of considering the projection map $\gamma T \to \alpha T$ between two Hausdorff compactifications $\gamma T \geqslant \alpha T$ of a locally compact non-compact Hausdorff space T. G.D. Faulkner has studied in [6] whether the extension

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1 Partially supported by DGICYT grants MTM2005-05207 and MTM2004-07665-C02-02, and by Junta de Castilla y León grant LE032A09.
2 Partially supported by DGICYT grant MTM2004-07665-C02-02 and by Junta de Extremadura grant GRU08040.

0166-8641/\$ – see front matter
doi:10.1016/j.topol.2009.10.009

$C(\alpha T) \subseteq C(\gamma T)$ has a primitive element. This extension has been also studied by T. Dwornik-Orzechowska and E. Wajch in [5].

For simplicity, we shall always assume that $\pi : X \to Y$ is an onto continuous map. Therefore, the induced morphism $C(Y) \to C(X)$ is injective and we shall consider $C(Y)$ as a subring of $C(X)$, i.e., we identify g with $g \circ \pi$, for every $g \in C(X)$.

The present paper is a continuation of [4]. We proved there that, in order to study the finiteness properties between the associated rings of continuous functions (i.e., whether the ring extension is finite, integral, singly generated or finitely generated), the problem for compactifications of locally compact Hausdorff spaces is equivalent to the problem for arbitrary compact Hausdorff spaces. It was also proved there that, for X and Y compact Hausdorff spaces, $C(X)$ is finitely generated as a $C(Y)$-module if and only if the map $X \to Y$ is locally injective. We prove here that if the extension $C(Y) \subseteq C(X)$ has a primitive element, then $C(X)$ is finitely generated as a $C(Y)$-module and, as a consequence, the map π is locally injective. For unbranched finite coverings $\pi : X \to Y$ we get a quite neat result: If the space Y is connected and the extension $C(Y) \subseteq C(X)$ has a primitive element, then π is a trivial covering.

Let t be an independent variable over $C(Y)$. For each function $f \in C(X)$ we consider the evaluation morphism

$$C(Y)[t] \to C(Y)[f],$$
$$P(t) \mapsto P(f).$$

The kernel of this morphism is called the *constraint ideal* of f and it is denoted by I_f. We show that if the space Y is connected and the constraint ideal of a primitive element f is principal, then this ideal is generated by a monic polynomial $(t - h_1) \cdots (t - h_n) \in C(Y)[t]$ and the space X decomposes into a finite family of pairwise disjoint zero-sets

$$X = Z(f - h_1) \cup \cdots \cup Z(f - h_n),$$

each homeomorphic to Y via π, so that $\pi : X \to Y$ is again a trivial covering. As usual, $Z(f)$ denotes the *zero-set* of f, that is, $Z(f) = \{x \in X\colon\ f(x) = 0\}$.

1. Preliminaries

For rings of continuous functions we use the same terminology as in [7]. For algebraic concepts the reader may consult [1]. Nevertheless we shall review some definitions that will be used throughout the paper.

1.1. Definition. Let $A \subseteq B$ be an extension of rings, and consider B with the induced structure as an A-module. Then B is said to be an *A-algebra*.

The extension $A \subseteq B$ is *finite*, and B is a *finite A-algebra*, if B is a finitely generated A-module, i.e., if there exists a finite set of elements $b_1, \dots, b_n \in B$ such that every element of B is a linear combination of $b_1, \dots, b_n$ with coefficients in A, i.e., $B = A \cdot b_1 + \cdots + A \cdot b_n$.

An element $b \in B$ is *integral* over A if there exists a monic polynomial $P(t) \in A[t]$ such that $P(b) = 0$. Of course we say that $P(t)$ is a *monic* polynomial if its leading coefficient is a unit of A.

The extension is *integral*, and B is an *integral A-algebra*, if every element of B is integral over A.

The extension is of *finite type*, and B is a *finitely generated* A-algebra, if there exists a finite set of elements $b_1, \dots, b_n \in B$ such that every element of B can be written as a polynomial in $b_1, \dots, b_n$ with coefficients in A, i.e., $B = A[b_1, \dots, b_n]$.

The extension is *singly generated*, and B is a *singly generated A-algebra*, if there exists an element $b \in B$ such that $B = A[b]$. In this case we say that b is a *primitive element*.

Recall that a ring extension is finite if and only if it is of finite type and integral.

If X is a Tychonoff space, we shall write j_X to denote the map that sends $x \in X$ to the maximal ideal $M_x = \{f \in C(X)\colon f(x) = 0\}$ of $C(X)$. We consider M_x as a point in the maximal spectrum $\operatorname{Max} C(X)$ of $C(X)$. It is well known (see [7, 4.9]) that if X is a compact Hausdorff space then j_X is a homeomorphism.

2. Monic polynomials with coefficients in $C(Y)$

Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces, and assume that the induced extension $C(Y) \subseteq C(X)$ has a primitive element f. The main purpose of this section is to prove that f is an integral element over $C(Y)$. The key tool is a classic result from complex variable theory, which we have taken directly from [7, 13.3], asserting that the roots of a polynomial depend continuously on its coefficients (see also [11]).

Let us quote this result. For each point

$$a = (a_1, \dots, a_n) \in \mathbb{R}^n,$$

let $\rho_1(a), \dots, \rho_n(a)$ denote the real parts of the (complex) zeros of the polynomial

$$t^n + a_1 t^{n-1} + \cdots + a_n$$

(listing each according to its multiplicity), indexed so that

$$\rho_1(a) \leqslant \cdots \leqslant \rho_n(a).$$

2.1. Theorem. *([7, 13.3(a)]) The functions $\rho_1, \ldots, \rho_n$ from $\mathbb{R}^n$ to $\mathbb{R}$ are continuous.*

For $P(t) = g_0 t^n + \cdots + g_{n-1}t + g_n \in C(Y)[t]$ and $y \in Y$, we write $P_y(t) = g_0(y)t^n + \cdots + g_{n-1}(y)t + g_n(y) \in \mathbb{R}[t]$.

2.2. Corollary. *Let Y be an arbitrary topological space, and set $Q(t) = t^n + g_1 t^{n-1} + \cdots + g_n \in C(Y)[t]$. Assume that, for every $y \in Y$, all the roots of $Q_y(t)$ are in $\mathbb{R}$. Then there exist functions $h_1, \ldots, h_n \in C(Y)$ such that $Q(t) = (t - h_1) \cdots (t - h_n)$.*

Proof. If $g(y) = (g_1(y), \ldots, g_n(y))$ then, with the notation in Theorem 2.1, the roots of $Q_y(t)$ are $\rho_1(g(y)), \ldots, \rho_n(g(y))$. Hence, $Q_y(t) = (t - \rho_1(g(y))) \cdots (t - \rho_n(g(y)))$. Since the last equality holds for every $y \in Y$, we conclude that

$$Q(t) = (t - \rho_1 \circ g) \cdots (t - \rho_n \circ g).$$

Thus, one can take $h_i = \rho_i \circ g$. □

2.3. Corollary. *Let $\pi : X \to Y$ be a surjective continuous map between arbitrary topological spaces, and let $f \in C(X)$. If f satisfies a monic polynomial of degree n with coefficients in $C(Y)$, then there exist functions $h_1, \ldots, h_n$ in $C(Y)$ such that $(f - h_1) \cdots (f - h_n) = 0$.*

Proof. We may assume without loss of generality that $f^n + g_1 \cdot f^{n-1} + \cdots + g_n = 0$, where $g_1, \ldots, g_n \in C(Y)$. Set $g = (g_1, \ldots, g_n) : Y \to \mathbb{R}^n$.

With the notation in Theorem 2.1,

$$f(x) \in \{\rho_1(g(\pi(x))), \ldots, \rho_n(g(\pi(x)))\},$$

for every $x \in X$.

Therefore, f satisfies the polynomial

$$(t - \rho_1 \circ g) \cdots (t - \rho_n \circ g). \qquad \square$$

2.4. Theorem. *Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. If the induced extension $C(Y) \subseteq C(X)$ has a primitive element, then it is an integral extension.*

Proof. First of all note that if the extension $C(Y) \subseteq C(X)$ has a primitive element f, then one may assume that $f \geqslant 1$: Since X is a compact space, then f has a lower bound and so there exists $m \in \mathbb{N}$ such that $-m \leqslant f$. Hence, $1 \leqslant f + (m+1)$, and certainly $C(Y)[f] = C(Y)[f + (m+1)]$. Therefore, we may replace f by $f + (m+1)$ if necessary.

From now on we assume that the extension $C(Y) \subseteq C(X)$ has a primitive element $f \geqslant 1$.

Since $1/f \in C(X) = C(Y)[f]$, then $1/f = -\sum_{i=1}^{n} g_i \cdot f^{i-1}$, where $n \in \mathbb{N}$ and $g_i \in C(Y)$. If we multiply both members by $1/f^{n-1}$ and then add them together, we get

$$\frac{1}{f^n} + g_1 \cdot \frac{1}{f^{n-1}} + \cdots + g_n = 0.$$

This shows that the function $1/f$ is an integral element over $C(Y)$.

Let us see that f is also integral over $C(Y)$. According to Corollary 2.3, there exist functions $h_1, \ldots, h_n$ in $C(Y)$ such that

$$\left(\frac{1}{f} - h_1\right) \cdots \left(\frac{1}{f} - h_n\right) = 0.$$

Since X is compact, there exists $\delta > 0$ such that $1/f \geqslant \delta$.

Finally,

$$\left(f - \frac{1}{h_1 \vee \delta}\right) \cdots \left(f - \frac{1}{h_n \vee \delta}\right) = 0.$$

Now it follows from [1, 5.1 and 5.2] that $C(X)$ is integral over $C(Y)$. □

Considering the projection map $\gamma T \to \alpha T$ between two Hausdorff compactifications $\gamma T \geqslant \alpha T$ of a locally compact non-compact Hausdorff space T, we have already mentioned that Faulkner has studied in [6] whether the extension $C(\alpha T) \subseteq C(\gamma T)$ has a primitive element. The next corollary is the deepest result about it.

2.5. Corollary. *Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. If the induced extension $C(Y) \subseteq C(X)$ has a primitive element, then the map π is locally injective.*

Proof. We have just seen that if the extension $C(Y) \subseteq C(X)$ has a primitive element, then it is an integral extension and hence a finite one. It is proved in [10, 5.6] that if the extension $C(Y) \subseteq C(X)$ is finite then the corresponding map $\pi : X \to Y$ is locally injective. □

2.6. Remark. Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. We do not know if every finitely generated extension $C(Y) \subseteq C(X)$ is finite, we do not even know the answer for the case $C(X) = C(Y)[f_1, f_2]$.

If the space Y is not connected, then a monic polynomial with coefficients in $C(Y)$ can be the product of two non-monic polynomials: Let Y be the union of two disjoint non-void open subsets Y_1 and Y_2. If $P(t) = g_0 t + 1$ and $Q(t) = h_0 t + 1$, where g_0 and h_0 are, respectively, the characteristic functions of Y_1 and Y_2, then $P(t)$ and $Q(t)$ are two non-monic polynomials whose product, $P(t)Q(t) = t + 1$, is a monic polynomial.

We are going to see that this situation cannot happen when the space Y is connected.

2.7. Theorem. *Let Y be a connected space and let $P(t), Q(t) \in C(Y)[t]$. The product $P(t)Q(t)$ is a monic polynomial if and only if both $P(t)$ and $Q(t)$ are monic polynomials.*

Proof. Assume that the product $P(t)Q(t)$ is a monic polynomial. We shall show that both $P(t)$ and $Q(t)$ are monic polynomials too.

Let $P(t), Q(t) \in C(Y)[t]$, $P(t) = g_0 t^n + g_1 t^{n-1} + \cdots + g_n$ and $Q(t) = h_0 t^m + h_1 t^{m-1} + \cdots + h_m$, with $g_0 \neq 0$ and $h_0 \neq 0$.

We may assume that

$$P(t)Q(t) = t^k + l_1 t^{k-1} + \cdots + l_k \in C(Y)[t]. \tag{1}$$

We want to prove that $k = n + m$. Assume, on the contrary, that $k = n + m - r$, with $r > 0$.

First we shall demonstrate that

$$g_0 h_0 = g_0 h_1 = \cdots = g_0 h_{r-1} = 0. \tag{2}$$

It follows from (1) that

$$\begin{cases} 0 = g_0 h_0, \\ 0 = g_0 h_1 + g_1 h_0, \\ \cdots \\ 0 = g_0 h_{r-1} + g_1 h_{r-2} + \cdots + g_{r-1} h_0. \end{cases} \tag{3}$$

It follows from the first equation in (3) that, if $g_0(y) \neq 0$, then $h_0(y) = 0$ and, taking into account the second equation in (3), one concludes that $h_1(y) = 0$. Thus, $g_0 h_1 = 0$. From the next equations in (3), and arguing in a similar way, we deduce that $g_0 h_2 = \cdots = g_0 h_{r-1} = 0$.

It also follows from (1) that

$$g_0 h_r + g_1 h_{r-1} + \cdots + g_r h_0 = 1. \tag{4}$$

Next we shall prove that the function $g_0 h_r$ can only take the values 0 and 1.

If $g_0(y) \neq 0$, then it follows from (2) that

$$h_0(y) = h_1(y) = \cdots = h_{r-1}(y) = 0.$$

Now one deduces from (4) that $g_0(y)h_r(y) = 1$. Therefore, $g_0(y)h_r(y) \in \{0, 1\}$, for every $y \in Y$.

Since Y is a connected space, either $g_0 h_r = 1$ or $g_0 h_r = 0$.

In the first case we conclude that $P(t)$ is a monic polynomial, and so the degree of $P(t)Q(t)$ is $n + m$, which contradicts our assumption.

In the second case we have

$$g_0 h_0 = g_0 h_1 = \cdots = g_0 h_r = 0.$$

If $g_0(y) \neq 0$, then $h_0(y) = h_1(y) = \cdots = h_r(y) = 0$ and, as a consequence,

$$g_0(y)h_r(y) + g_1(y)h_{r-1}(y) + \cdots + g_r(y)h_0(y) = 0,$$

which contradicts (4). Hence $g_0(y) = 0$, for every $y \in Y$. This is also a contradiction. □

2.8. Remark. As the referee has pointed out to us, the argument in Theorem 2.7, with slight changes, also works if one replaces $C(Y)$ for any reduced (= semiprime) commutative ring A with no idempotent other than 0 and 1. This latter condition is equivalent to the prime spectrum of A being a connected space. Although the result is probably known, we have provided a proof for the sake of completeness.

3. When the constraint ideal is principal: the case of trivial coverings

Next we shall adapt some definitions in [10] to the compact case.

To stress that X is the union of a family of pairwise disjoint subsets $(X_i)_{i=1}^n$, we shall write $X = X_1 \coprod \cdots \coprod X_n$.

3.1. Definition. Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. We say that π is a *trivial covering* if the space X has a cover by pairwise disjoint open subsets each of them homeomorphic to Y via π. We say that π is a *quasi-trivial covering* if it decomposes into several trivial coverings, i.e., if the space Y is the union of a finite family of pairwise disjoint open subsets $Y = Y_1 \coprod \cdots \coprod Y_n$ and $\pi : \pi^{-1}(Y_i) \to Y_i$ is a trivial covering for every i. We say that π is an *unbranched covering* if every point of Y has a neighborhood U such that $\pi : \pi^{-1}(U) \to U$ is a trivial covering. The map π is said to be *finite* if each fiber $\pi^{-1}(y)$ is a finite set, and it is said to be a *finite covering* if it is a finite and open map.

3.2. Remark. Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. Assume that the induced extension $C(Y) \subseteq C(X)$ has a primitive element $f \in C(X)$. Given two different points x_1, x_2 in the same fiber $\pi^{-1}(y)$, there exists $l \in C(X)$ such that $l(x_1) \neq l(x_2)$. Since all the functions in $C(X)$ are polynomial expressions in f with coefficients in $C(Y)$, and the functions in $C(Y)$ are constant on each fiber of π, then the above inequality can only hold if $f(x_1) \neq f(x_2)$. One concludes that the primitive element f is injective on each fiber of π. To see that the converse does not hold, take $Y = [0,1] \subseteq \mathbb{R}$ and $X = Y \times Y$, and denote by $\pi_1 : X \to Y$ and $\pi_2 : X \to \mathbb{R}$, respectively, the first and the second coordinate projection. Certainly π_2 is injective on each fiber of π_1, but, according to Corollary 2.5, it is not a primitive element for the extension $C(Y) \subseteq C(X)$ induced by π_1, since the map π_1 is not locally injective. Nevertheless, we are going to prove in Theorem 3.3 that the converse does hold in the case of a trivial covering, and we shall see in the proof of Theorem 3.11 that the same happens for finite coverings.

If $\pi : X = \coprod_{i=1}^n X_i \to Y$ is a trivial covering, where each X_i is homeomorphic to Y via π, we shall denote by η_i the inverse map of $\pi : X_i \to Y$.

3.3. Theorem. *Let $\pi : X = \coprod_{i=1}^n X_i \to Y$ be a trivial covering between compact Hausdorff spaces, where each X_i is homeomorphic to Y via π. If $f \in C(X)$ is injective on each fiber $\pi^{-1}(y)$, then f is a primitive element for the extension $C(Y) \subseteq C(X)$, and the constraint ideal of f is the principal ideal of $C(Y)[t]$ generated by $(t-h_1)\cdots(t-h_n)$, where $h_i = f|_{X_i} \circ \eta_i \in C(Y)$.*

Proof. Assume that f is injective on each fiber of π. First we shall see that $C(X) = C(Y)[f]$.

We define

$$f_i = \frac{\prod_{i\neq j}(f-h_j)}{\prod_{i\neq j}(h_i-h_j)}, \quad \text{for } 1 \leqslant i \leqslant n.$$

Certainly $f_i \in C(Y)[f]$. Moreover, $f_i(X_i) = \{1\}$ and $f_i(X_j) = \{0\}$ for $i \neq j$, that is, f_i is the characteristic function on X_i. Now observe that, for any $l \in C(X)$, $l = \sum_{i=1}^n (l|_{X_i} \circ \eta_i)\cdot f_i$, and so $l \in C(Y)[f]$.

Next we shall prove that I_f is the principal ideal of $C(Y)[t]$ generated by $Q(t) = (t-h_1)\cdots(t-h_n)$.

Clearly $Q(f) = 0$. Assume now that $P(f) = 0$, for some $P(t) \in C(Y)[t]$. We want to see that $P(t)$ is a multiple of $Q(t)$.

Since $Q(t)$ is a monic polynomial, then $P(t) = Q(t)S(t) + R(t)$, for some $S(t)$ and $R(t)$ in $C(Y)[t]$, with $\deg R(t) < n$ (see [8, 4.1 of Chapter V]). Hence $R(f) = 0$. We are going to show that $R(t) = 0$.

For any $y \in Y$, the values $h_1(y), \dots, h_n(y)$ are different to each other, since $f(\pi^{-1}(y)) = \{h_1(y), \dots, h_n(y)\}$, and they all satisfy the polynomial $R_y(t)$. Since $\deg R_y(t) < n$, then $R_y(t) = 0$, for every $y \in Y$. Therefore, $R(t) = 0$, and so $P(t)$ is a multiple of $Q(t)$. $\square$

3.4. Remark. We shall show in Theorem 3.7 that the previous result cannot be extended to the case when $\pi : X \to Y$ is a quasi-trivial covering. Nevertheless it is not difficult to see that if $\pi : X \to Y$ is a quasi-trivial covering and $f \in C(X)$ is a function that is injective on each fiber of π, then f is a primitive element for the induced extension $C(Y) \subseteq C(X)$. Next we shall prove that the constraint ideal of f is a principal one.

Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. Suppose that Y is the union of several pairwise disjoint non-void open subsets, $Y = Y_1 \coprod \cdots \coprod Y_n$. Set $X_i = \pi^{-1}(Y_i)$ and assume that $\pi : X_i \to Y_i$ is a trivial covering for any $1 \leqslant i \leqslant n$. Let $X_i = X_i^1 \coprod \cdots \coprod X_i^{m_i}$, where each $\pi : X_i^j \to Y_i$ is a homeomorphism. Assume that $m_j \neq m_k$ for some $1 \leqslant j,k \leqslant n$, so that $\pi : X \to Y$ is not a trivial covering.

Let $f \in C(X)$ be a function that is injective on each fiber of π, and let f_i be the restriction of f to X_i. According to the preceding theorem, $C(X_i) = C(Y_i)[f_i]$, and I_{f_i} is the principal ideal of $C(Y_i)[t]$ generated by a monic polynomial $Q_i(t)$.

Since $C(X) = C(X_1) \times \cdots \times C(X_n)$ and $C(Y) = C(Y_1) \times \cdots \times C(Y_n)$, it follows from the Chinese remainder theorem that $I_f = (\tilde{Q}_1(t), \dots, \tilde{Q}_n(t))$, where $\tilde{Q}_i(t) \in C(Y)[t]$ is the polynomial one obtains by extending by zero to the whole space Y the coefficients of $Q_i(t) \in C(Y_i)[t]$.

Note that $(\tilde{Q}_1(t),\dots,\tilde{Q}_n(t))$ is a principal ideal, since $(\tilde{Q}_1(t),\dots,\tilde{Q}_n(t)) = (\tilde{Q}_1(t)+\cdots+\tilde{Q}_n(t))$, but the generator $\tilde{Q}_1(t)+\cdots+\tilde{Q}_n(t)$ is not a monic polynomial. And we shall see in Theorem 3.7 that the ideal $(\tilde{Q}_1(t)+\cdots+\tilde{Q}_n(t))$ cannot have any monic generator.

Now we establish a result concerning ideals and continuous maps that we shall use to prove Theorem 3.7.

Let $h: A \to B$ be a ring morphism and let $\mathfrak{a}$ be an ideal of A. We shall denote by $\mathfrak{a}\cdot B$ the extension of $\mathfrak{a}$ to the ring B, that is,

$$\mathfrak{a}\cdot B = \Big\{\sum h(a_i)b_i\colon\ a_i\in\mathfrak{a},\ b_i\in B\Big\}.$$

3.5. Lemma. *Let $\pi : X \to Y$ be a continuous map between Tychonoff spaces.*

(a) *If I_F is the ideal of those functions in $C(Y)$ that vanish on a closed subset $F\subseteq Y$, then $I_F\cdot C(X)$ is a radical ideal in $C(X)$, that is,*

$$I_F\cdot C(X) = \operatorname{rad}\big(I_F\cdot C(X)\big) = \big\{f\in C(X)\colon\ f^n\in I_F\cdot C(X)\ \textit{for some}\ n>0\big\}.$$

(b) *If X is a real-compact space and the induced morphism $C(Y)\to C(X)$ is integral, then, for every $y\in Y$,*

$$M_y\cdot C(X) = I_{\pi^{-1}(y)} = M_{x_1}\cap\cdots\cap M_{x_n},$$

where $\{x_1,\dots,x_n\} = \pi^{-1}(y)$.

Proof. (a) To prove that $\operatorname{rad}(I_F\cdot C(X))$ is contained in $I_F\cdot C(X)$, it is enough to prove that if $f\geqslant 0$ and $f^2\in I_F\cdot C(X)$ then $f\in I_F\cdot C(X)$.

Suppose that $f^2=\sum_{i=1}^n g_i f_i$, where $g_i\in I_F$ and $f_i\in C(X)$. There exists $g\in C(Y)$ such that g^2 divides every g_i and $Z(g)=Z(g_1)\cap\cdots\cap Z(g_n)$ (see Lemma 2.1 in [9]). Therefore, $f^2=g^2 l$, where $g\in I_F$ and $l\in C(X)$. By replacing l by $l\vee 0$, if necessary, we can assume that $l\geqslant 0$. Thus, $f=|g|\sqrt{l}\in I_F\cdot C(X)$.

(b) We are going to prove that $\operatorname{rad}(M_y\cdot C(X)) = I_{\pi^{-1}(y)}$. Then the result will follow from (a).

The radical of $M_y\cdot C(X)$ is the intersection of all the prime ideals containing it [1, 1.14]. Let h denote the ring morphism $C(Y)\to C(X)$ induced by π. If $\mathfrak{p}$ is a prime ideal in $C(X)$ containing $M_y\cdot C(X)$, then $h^{-1}(M_y\cdot C(X)) = M_y\subseteq h^{-1}(\mathfrak{p})$, so that $h^{-1}(\mathfrak{p}) = M_y$. By hypothesis, the morphism $C(Y)\to C(X)$ is integral. This implies that $\mathfrak{p}$ is a maximal ideal in $C(X)$ and that the morphism $C(Y)/M_y=\mathbb{R}\to C(X)/\mathfrak{p}$ is also integral [1, 5.8 and 5.6]. Now it follows that $C(X)/\mathfrak{p}=\mathbb{R}$, because, for every prime ideal $\mathfrak{p}$ in $C(X)$, the quotient ring $C(X)/\mathfrak{p}$ totally ordered [7, 5.5]. Then $\mathfrak{p}=M_{x_i}$ for some $1\leqslant i\leqslant n$, since X is a real-compact space and the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "\pi"'] & \operatorname{Max} C(X) \arrow[d, "\pi_\beta"] \\ Y \arrow[r, "j_Y"'] & \operatorname{Max} C(Y) \end{tikzcd}$$

where $\pi_\beta(M) = h^{-1}(M)$. $\square$

3.6. Remark. Lemma 3.5(b) can also be obtained as a consequence of Corollary 2.5(i) in [9], which states: "If the radical of an ideal I in $C(X)$ is a z-ideal then $I=\operatorname{rad}(I)$". See also [2, 2.4].

3.7. Theorem. *Let $\pi : X\to Y$ be a surjective continuous map between compact Hausdorff spaces. The induced extension $C(Y)\subseteq C(X)$ has a primitive element whose constraint ideal is generated by a monic polynomial if and only if $\pi : X\to Y$ is a trivial covering.*

Proof. Suppose that $C(X)=C(Y)[f]$ and that $I_f=(Q(t))$, where $Q(t)$ is a monic polynomial of degree n with coefficients in $C(Y)$. We may assume without loss of generality that the leading coefficient of $Q(t)$ is 1.

It follows from Corollary 2.3 that there exist functions $h_1,\dots,h_n$ in $C(Y)$ such that f is a root of the polynomial $S(t)=(t-h_1)\cdots(t-h_n)$. Both $S(t)$ and $Q(t)$ are monic polynomials of the same degree and the same leading coefficient. Certainly, $S(t)$ is a multiple of $Q(t)$. Hence $Q(t)=S(t)$.

Next we shall show that for $i\neq j$, $h_i(y)\neq h_j(y)$ for every $y\in Y$. Then, $X=Z(f-h_1)\coprod\cdots\coprod Z(f-h_n)$, and we shall finally see that each $Z(f-h_i)$ is homeomorphic to Y via π.

Let y be a point in Y. On the one hand, since $C(X)\cong C(Y)[t]/(Q(t))$, then

$$\frac{C(X)}{M_y\cdot C(X)} \simeq \frac{(\frac{C(Y)}{M_y})[t]}{(Q_y(t))} = \frac{\mathbb{R}[t]}{((t-h_1(y))\cdots(t-h_n(y)))}.$$

On the other hand, by Lemma 3.5(b), $C(X)/(M_y\cdot C(X)) = C(X)/I_{\pi^{-1}(y)}$ and, by Tietze's extension theorem (see [12, 15.8] or [7]), $C(X)/I_\pi^{-1}(y)$ is isomorphic to $C(\pi^{-1}(y))$. This last ring, as any ring of continuous functions, does not have non-zero nilpotent elements, so that the same is true for $\mathbb{R}[t]/(Q_y(t))$, and so the polynomial $Q_y(t) = (t - h_1(y)) \cdots (t - h_n(y))$ does not have multiple roots, that is, $h_i(y) \neq h_j(y)$ for $i \neq j$. Thus, $X = Z(f - h_1) \coprod \cdots \coprod Z(f - h_n)$.

Moreover, from the isomorphism

$$C\big(\pi^{-1}(y)\big) \cong \frac{\mathbb{R}[t]}{((t - h_1(y)) \cdots (t - h_n(y)))} = \mathbb{R} \times \cdots \times \mathbb{R},$$

it follows that $\pi^{-1}(y)$ has exactly n different points. Consequently, the continuous map $\pi : Z(f - h_i) \to Y$ is one-to-one and onto and, since both $Z(f - h_i)$ and Y are compact Hausdorff spaces, it is a homeomorphism. $\square$

3.8. Remark. As regards to the proof of the preceding theorem, note that once we have shown that the graphs of the functions h_i are pairwise disjoint (for $i \neq j$, $h_i(y) \neq h_j(y)$ for every $y \in Y$), we can assert that the principal ideals $(t - h_i)$ of $C(Y)[t]$ are pairwise comaximal, that is, the ideal generated by any two of them is the whole ring. The last part of the proof is equivalent to apply the Chinese remainder theorem and conclude that $C(Y)[t]/(Q(t))$ is isomorphic to the product ring

$$\frac{C(Y)[t]}{(t - h_1)} \times \cdots \times \frac{C(Y)[t]}{(t - h_n)}.$$

This product ring is, in turn, isomorphic to $C(Y) \times \cdots \times C(Y)$ and, finally,

$$X \cong \operatorname{Max} \frac{C(Y)[t]}{(Q(t))} \cong Y \coprod \cdots \coprod Y.$$

3.9. Theorem. *Let $\pi : X \to Y$ be a surjective continuous map between compact Hausdorff spaces. Assume that Y is a connected space and that $C(X) = C(Y)[f]$. If the constraint ideal of f is a principal ideal, then it is generated by a monic polynomial and, as a consequence, $\pi : X \to Y$ is a trivial covering.*

Proof. Suppose that $I_f = (P(t))$. By Theorem 2.4, f is an integral element over $C(Y)$ and so there is a monic polynomial $Q(t) \in C(Y)[t]$ such that $Q(f) = 0$. Therefore, $Q(t) = P(t)S(t)$, for some polynomial $S(t) \in C(Y)[t]$. Since $Q(t)$ is a monic polynomial, it follows from Theorem 2.7 that $P(t)$ is monic too. $\square$

3.10. Example. Set $S^1_+ = \{z = (x, y) \in S^1 \colon\ y \geqslant 0\}$ and let $\pi : S^1_+ \to S^1 \subseteq \mathbb{C}$ be the map defined by $\pi(z) = z^2$. We know from [4, 4.1 and 3.3] that the induced extension $C(S^1) \subseteq C(S^1_+)$ has a primitive element f. Nevertheless, since $\pi : S^1_+ \to S^1$ is not a trivial covering, it follows from the preceding theorem that the ideal I_f is not principal.

We shall write $|S|$ to denote the cardinality of the set S.

3.11. Theorem. *Let $\pi : X \to Y$ be a finite covering of compact Hausdorff spaces. If the induced extension $C(Y) \subseteq C(X)$ has a primitive element, then $\pi : X \to Y$ is a quasi-trivial covering. Consequently, if we also assume that Y is a connected space then $\pi : X \to Y$ is a trivial covering.*

Proof. If the extension $C(Y) \subseteq C(X)$ has a primitive element, then, according to Corollary 2.5, π is a locally injective map. Therefore, the cardinality of the fibers of π is constant on some neighborhood of each point $y \in Y$, and Y is the union of the pairwise disjoint open and closed sets $Y_i = \{y \in Y \colon\ |\pi^{-1}(y)| = i\}$. Moreover, if f is a primitive element for the extension $C(Y) \subseteq C(X)$, then $f_i = f|_{Y_i}$ is a primitive element for $C(Y_i) \subseteq C(\pi^{-1}(Y_i))$. Thus, it is enough to prove that if all the fibers of π have exactly n points then π is a trivial covering.

Let f be a primitive element for the extension $C(Y) \subseteq C(X)$. Although Theorem 2.4 assures us that f satisfies a monic polynomial with coefficients in $C(Y)$, it does not inform us about the degree of that polynomial. For our argument to work, we do need a monic polynomial of degree n. Le us repeat the argument in [10, 2.8] to find the monic polynomial of the right degree.

Let $S_1(f)(y), \ldots, S_n(f)(y)$ be the elementary symmetric polynomials on the values $f(x_1), \ldots, f(x_n)$, where $\{x_1, \ldots, x_n\} = \pi^{-1}(y)$.

The functions $S_i(f) : Y \to \mathbb{R}$ are continuous and f is a root of the polynomial

$$t^n - S_1(f)t^{n-1} + \cdots + (-1)^n S_n(f).$$

By Corollary 2.3, there are functions $h_1, \ldots, h_n$ in $C(Y)$ such that

$$(f - h_1) \cdots (f - h_n) = 0.$$

Hence,

$$X = Z(f - h_1) \cup \cdots \cup Z(f - h_n).$$

Since $f(\pi^{-1}(y)) \subseteq \{h_1(y), \dots, h_n(y)\}$ and f is injective on $\pi^{-1}(y)$, one concludes that the values $h_1(y), \dots, h_n(y)$ are different to each another and so $f(\pi^{-1}(y)) = \{h_1(y), \dots, h_n(y)\}$.

Thus,

$$X = Z(f - h_1) \coprod \cdots \coprod Z(f - h_n).$$

It is now clear that $\pi : Z(f - h_i) \to Y$ is a one-to-one and onto continuous map. Moreover, taking into account that both $Z(f - h_i)$ and Y are compact Hausdorff spaces, one concludes that $\pi : Z(f - h_i) \to Y$ is a homeomorphism. $\square$

3.12. Remark. Let $\pi : X \to Y$ be an unbranched finite covering of compact Hausdorff spaces. The preceding argument shows that if there is a function in $C(X)$ that is injective on each fiber of π, then $\pi : X \to Y$ is a quasi-trivial covering.

3.13. Example. Let $\pi : S^n \to \mathbb{P}^n$ be the natural projection map from the unit sphere onto the real projective space, $\pi(x_1, \dots, x_{n+1}) = \pi(-x_1, \dots, -x_{n+1})$. Since π is an unbranched finite covering, the induced extension $C(\mathbb{P}^n) \subseteq C(S^n)$ is finite, but it follows from the preceding theorem that it does not have a primitive element. It also follows that, for every continuous function f in $C(S^n)$, there exists a point $p \in S^n$ such that $f(p) = f(-p)$, because, if there were no such point in S^n, then f would be injective on each fiber of π.

Acknowledgement

The authors would like to thank the referee for the careful reading of this paper and for pointing out that the argument in Theorem 2.7 is essentially point-free.

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